GALOIS THEORY FOR A CLASS OF MODULAR LATTICES

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Abstract. We construct Galois theory for sublattices of certain complete modular lattices and their automorphism groups. A well–known description of the intermediate subgroups of the general linear group over an Artinian ring containing the group of diagonal matrices, due to Z.I.Borewicz and N.A.Vavilov, can be obtained as a consequence of this theory. Bibliography: 11 titles.

§ 1. Introduction

The description of subgroups in the general linear group over a semilocal ring R containing the group of diagonal matrices was obtained in the series of papers of Z.I.Borewicz and N.A.Vavilov [Bo2], [BV], [V1], [V2]. One may find a wealth of background information and many further related references in the surveys [V3], [V4].

A.Z.Simonian [S] showed how to state these results in terms of Galois correspondences between sublattices of certain lattices and subgroups of their automorphism groups for the case, when R is a field.

Galois theory for lattices is constructed in the present paper. The description of the intermediate subgroups in the general linear group over an Artinian ring, containing the group of diagonal matrices, can be deduced from the results proved here.

Let L be a lattice and G a subgroup of the group $\mathrm{Aut}(L)$ of all automorphisms of the lattice L. Consider a subgroup F of the group G and a sublattice M of the lattice L. By definition, put

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$L(F) = \{l \in L$ such that $f(l) = l$ for every $f \in F\}$,

$G(M) = \{g \in G$ such that $g(m) = m$ for every $m \in M\}$

(it is clear that $L(F)$ is a sublattice of L and $G(M)$ is a subgroup of G).

Let L_0 be a sublattice of L; we put $H = G(L_0)$, $L_0^{'} = L(H)$. The set of sublattices of $L_0^{'}$ is denoted by $\mathfrak{M}$ and the set of subgroups of G containing H by $\mathfrak{N}$. We define mappings $\varphi : \mathfrak{M} \to \mathfrak{N}$ as $\varphi(M) = G(M)$ for $M \in \mathfrak{M}$ and $\psi : \mathfrak{N} \to \mathfrak{M}$ as $\psi(F) = L_0^{'}(F)$ for $F \in \mathfrak{N}$. It's easy to see that (φ, ψ) is a Galois correspondence between $\mathfrak{M}$ and $\mathfrak{N}$.

We denote hereafter the operations "infimum" and "supremum" in an arbitrary lattice as $\cdot$ and $+$, correspondingly.

If M is a lattice, $x_1, \dots, x_s \in M$, then for every i, $1 \leqslant i \leqslant s$, we put $\widehat{x}_i = x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_s$.

§ 2. Formulation of the main result

Let L be a modular lattice of finite length, L_0 its sublattice, which is a Boolean algebra, G a subgroup of the group of all automorphisms of the lattice L, $H = G(L_0)$.

Let $e_1, e_2, \dots, e_n$ be the atoms of L_0, $d(-)$ the dimension function on the lattice L. We require that the following conditions are fulfilled (it is supposed that, unless otherwise stated, the values of all indices are changing from 1 to n):

1^0. $0_{L_0} = 0_L$, $1_{L_0} = 1_L$.

2^0. The function d is constant on the set of atoms of L_0; we denote its value by m.

If there are two collections of elements in L, namely, $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$, then we write $(x_1, \dots, x_k) \leqslant (y_1, \dots, y_k)$, if $x_i \leqslant y_i$ for every i, $1 \leqslant i \leqslant k$. We define also the "infimum" and "supremum" of two such collections coordinatewise.

For every $x \in L$ its support $[x]$ is defined as the minimal (with respect to the ordering introduced above) collection $(x_1, \dots, x_n)$, where

$$x_i \leqslant e_i \quad \textit{and} \quad x \leqslant x_1 + \ldots + x_n \tag{+}$$

It is proved in § 4 that the support is well defined. We put $[x] = ([x]_1, \dots, [x]_n)$.

Let's denote by H_i the set of automorphisms of H which do not change all elements $x \in L$ such that $[x]_i = 0$.

For every $i \neq j$ and every $x \leqslant e_j$ we denote by $H_{ij}(x)$ the set of $f \in G$ such that:

1) $f(x_s) = x_s$ for every $s \neq i$, $x_s \leqslant e_s$

2) $[f(x_i)]_i = x_i$ for every $x_i \leqslant e_i$

3) $[f(e_i)]_k = \begin{cases} 0, & k \neq i, j; \\ e_i, & k = i; \\ x, & k = j \end{cases}$

(note that $H_{ij}(x)$ may be empty). Elements of $H_{ij}(x)$ will be called transvections.

We denote by $\overline{L}_0$ the set of elements of the form $\sum\limits_{i=1}^{n} x_i$, where $x_i \leqslant e_i$. It will be proved (see § 4) that it follows from the already imposed conditions that $\overline{L}_0$ is a sublattice of L.

We require that the following additional conditions are fulfilled:

3^0. If $a \in G$ and $[a(e_i)]_i = e_i$ for some i, then there exists $h \in H_i$ such that $[ha(x_i)]_i = [ah(x_i)]_i = x_i$ for every $x_i \leqslant e_i$.

4^0. There exists $h \in H_t \cap G(\overline{L}_0)$ such that $[aha^{-1}(x_i)]_r = [a([a^{-1}(x_i)]_t)]_r$ for every $a \in G,\ r \neq i,\ x_i \leqslant e_i$.

5^0. Let $u \in \overline{L}_0,\ u \geqslant e_i$ for some i; $g \in G,\ [g(u)]_i = e_i$. Then there exists $t \in G$ such that:

1) $[gt(e_i)]_i = e_i$,
2) $t(e_s) = e_s$ for every $s \neq i$,
3) $[t(e_i)]_j \leqslant [u]_j$.

6^0. If $f, g \in G$ are such that $[f(e_i)]_j \leqslant [g(e_i)]_j$ for some i, j, then $[f(x)]_j \leqslant [g(x)]_j$ for every $x \in L_0^{'},\ x \leqslant e_i$.

7^0. If $u \leqslant e_j$ for some j, then for every $i \neq j$ there exist $y_1 \leqslant e_j, \dots, y_s \leqslant e_j$ such that $u = \sum\limits_{r=1}^{s} y_r$ and $H_{ij}(y_r) \neq \varnothing$.

8^0. If $x = [f(e_i)]_j$ for some $f \in G,\ i \neq j$, then there exists $g \in H_{ij}(x)$ such that $[g(u)]_j = [f(u)]_j$ for every $u \leqslant e_i$.

9^0. If $w \in L,\ d(w) = m,$ and $[w] = (0, \dots, e_i, \dots, x, \dots, 0)$, where $H_{ij}(x) \neq \varnothing$, then there exists $t \in H_{ij}(x)$ such that $t(w) = e_i$.

10^0. Let $a_1 \in H_{ij}(x_1), \dots, a_s \in H_{ij}(x_s)$ and $y \leqslant x_1 + \ldots + x_s$ be such that $H_{ij}(y) \neq \varnothing$. Then $H_{ij}(y) \subseteq \langle H, a_1, \dots, a_s \rangle$.

11^0. If $a \in G$, then for every $t,\ i \neq j$ and every $h \in H_t$ the set $H_{ij}([aha^{-1}(e_i)]_j) \cap \langle a, H \rangle$ is not empty.

Theorem 2.1. *For every subgroup $F \geqslant H$ of the group G there exists a sublattice K of $L_0^{'}$ such that $G(K) \trianglelefteq F$. Moreover, if it is assumed that the lattice $\overline{L}_0(H)$ is finite, then the index $(F : G(K))$ is finite.*

Remark. The lattice K is not uniquely determined (see § 10).

§§ 4–8 are devoted to the proof of Theorem 2.1. The case $n = 1$ is trivial, therefore we assume hereafter that $n \geqslant 2$.

§ 3. The case $m = 1$

A.Z.Simonian [S] investigated the Galois correspondence introduced in § 1 for $m = 1$.

Theorem 2.1 and results of § 10 on uniqueness imply

Theorem 3.1. *Let L be a modular lattice of finite length, L_0 its sublattice of the same length which is a Boolean algebra, G a subgroup of the group of all automorphisms of the lattice L, $H = G(L_0)$, $L_0 = L(H)$. Provided that the conditions $1^{'} - 4^{'}$ stated below are fulfilled, for every subgroup $F \geqslant H$ of the group G there exists a unique sublattice K of the lattice L_0 containing 0 and 1 such that $G(K) \trianglelefteq F$ and $(F : G(K)) < \infty$.*

$1^{'}$. *There exists at least one automorphism from H_i, which changes all atoms $x \in L \setminus \{e_i\}$ such that $[x]_i = e_i$.*

$2^{'}$. *If $x, y \in L$ are atoms with $[x] = [y]$, then there exists $h \in H$ such that $h(x) = y$.*

$3^{'}$. *For every $i \neq j$ the set $H_{ij}(e_j)$ is not empty.*

$4^{'}$. *If $a \in G$, then for every t, $i \neq j$ and every $h \in H_t$ the set $H_{ij}([aha^{-1}(e_i)]_j) \cap \langle a, H\rangle$ is not empty.*

Note that the conditions $1^{'} - 4^{'}$ of Theorem 3.1 are not identical with the conditions of Theorem 2.1 [S], which seem to be simpler than ours.

§ 4. Properties of the support

Lemma 4.1. *If M is an arbitrary modular lattice, $x, y, z, t \in M$ and $(x+z)\cdot(y+t) = 0$, then $(x + y) \cdot (z + t) = x \cdot z + y \cdot t$.*

Proof. See [Bi].

Corollary 1. *If $x, y \in \overline{L}_0$, then $x \cdot y = \sum\limits_{i=1}^{n} [x]_i \cdot [y]_i$.*

Corollary 2. *$\overline{L}_0$ is a lattice.*

If there are two collections satisfying the condition (+) from § 2, then so does their "infimum". Since there exists at least one collection with the property (+) (see the condition 1^0), we see that the support is well defined.

Lemma 4.2. *If M is an arbitrary modular lattice, $x, x_1, \dots, x_s \in M$, then $\sum\limits_{i=1}^{s} (x + \widehat{x}_i) \cdot x_i = (\sum\limits_{i=1}^{s} x_i) \cdot \prod\limits_{i=1}^{s} (x + \widehat{x}_i)$.*

Proof. By induction, using the modularity law.

Lemma 4.3. *For every $x \in L$ and every i $[x]_i = (x + \widehat{e}_i) \cdot e_i$.*

Proof. By Lemma 4.2 $\sum\limits_{i=1}^{n} (x + \widehat{e}_i) \cdot e_i = \prod\limits_{i=1}^{n} (x + \widehat{e}_i) \geqslant x$. Further, $(x + \widehat{e}_i) \cdot e_i \leqslant ([x]_i + \widehat{e}_i) \cdot e_i = [x]_i$, and we get the desired equality.

Corollary. *For every $x, y \in L$ $[x + y] = [x] + [y]$.*

Lemma 4.4. *Let $v \in L$, $I \subseteq \{1,\dots,n\}$. Then there exists $v_I \in L$ such that $[v_I]_i = 0$ for $i \in I$, $v + \sum\limits_{i\in I}[v]_i = v_I + \sum\limits_{i\in I}[v]_i$.*

Proof. We put $v_I = (v + \sum\limits_{i\in I}[v]_i)\cdot(\sum\limits_{i\notin I}[v]_i)$. It is clear that $v_I + \sum\limits_{i\in I}[v]_i \leqslant v + \sum\limits_{i\in I}[v]_i$. It is easy to verify that the dimensions of the left-hand and right-hand parts coincide.

Corollary 1. *For every $t \in H_{ij}(x)$:*

(a) $t(e_i) + e_i = e_i + x$;

(b) $t(e_i) + x = e_i + x$.

Corollary 2. *If $t \in H_{ij}(x)$, then also $t^{-1} \in H_{ij}(x)$.*

§ 5. The auxiliary assertions

Hereafter by F we denote a subgroup of the group G containing the group H.

Lemma 5.1. *For every $a \in F$, indices t, $i \neq j$, and every $h \in H_t$ the set $H_{ij}([aha^{-1}(e_i)]_j) \cap F$ is not empty.*

Proof. Follows from the condition 11^0.

Lemma 5.2. *Let $u \in L$ be such that $d(u) = m$, $[u]_i = e_i$, $[u]_j = x$ (for some $i \neq j$), $H_{ij}(x) \neq \varnothing$. Then there exists $t \in H_{ij}(x)$ such that $[t(u)]_j = 0$.*

Proof. Let $i = 1$, $j = 2$. By Lemma 4.4 there exists $w \in L$ with the following properties: $[w] = (e_1, x, 0, \dots, 0)$; $u \leqslant w + w_1$, where $w_1 = [u]_3 + \dots + [u]_n$; $d(w) = m - d(u \cdot w_1)$. By Lemma 4.4 $u + \widehat{e}_1 = 1$. We have $d(u) = m$, therefore $u \cdot \widehat{e}_1 = 0$, whence $d(w) = m$. By the condition 9^0 there exists $t \in H_{12}(x)$ such that $t(w) = e_1$. We have $t(u) \leqslant t(w) + t(w_1) \leqslant \widehat{e}_2$, whence $[t(u)]_2 = 0$.

Lemma 5.3. *For every $h \in H$ there exist $h_i \in H_i$, $i = 1,\dots,n$ such that $h_n h_{n-1} \dots h_1 h \in G(\overline{L}_0)$.*

Proof. One must apply the condition 3^0.

We define for every $i \neq j$ "the ideals of transvections"

$$\sigma_{ij} = \sigma_{ij}(F) = \sum_{x:\ H_{ij}(x)\cap F\neq\varnothing} x$$

We also agree that $\sigma_{ii} = e_i$. Note that $\sigma_{ij} \leqslant e_j$ for every i, j.

Lemma 5.4. *(i) if $H_{ij}(x) \cap F \neq \varnothing$, then $H_{ij}(x) \subseteq F$.*

(ii) for every $y \leqslant \sigma_{ij}$ such that $H_{ij}(y) \neq \varnothing$ we have $H_{ij}(y) \subseteq F$.

Proof. One must apply the condition 10^0.

Lemma 5.5. *$\sigma_{ij} \in L_0^{'}$ for every i, j.*

Proof. Let $i \neq j$, $h \in H$. For L is a lattice of finite length, it is sufficient to show that $h(\sigma_{ij}) \leqslant \sigma_{ij}$. Let $f \in H_{ij}(x) \cap F$. Further, applying the Corollary 1(a) to Lemma 4.4, we obtain $[hf(e_i)]_j = h(x)$. By the condition 3^0 it is possible to find $\bar{h} \in H$ such that $hf\bar{h} \in H_{ij}(h(x))$. Since $hf\bar{h} \in F$, we have $h(x) \leqslant \sigma_{ij}$, hence $h(\sigma_{ij}) \leqslant \sigma_{ij}$.

We denote by $K = K(F)$ the sublattice of the lattice $\overline{L}_0$, generated by zero and elements $\sum\limits_{j=1}^{n} \sigma_{ij}$, where i changes from 1 to n. By Lemma 5.5 K is a sublattice of $L_0^{'}$.

§ 6. Proof of the inclusion $K \subseteq \overline{L_0(F)}$

We denote by $\overline{L_0(F)}$ the lattice which consists of the elements $l \in \overline{L}_0$ such that $f(l) \in \overline{L}_0$ for every $f \in F$.

Lemma 6.1. *Let $a \in F$. Let's put $u_s = \sum\limits_{j=1}^{n} [a(\sigma_{ij})]_s$ for every s. Then $[a^{-1}(u_s)] \leqslant (\sigma_{i1}, \ldots, \sigma_{in})$ for every s.*

Proof. By the definition,

$$[a^{-1}(u_s)] = [\sum_{j=1}^{n} a^{-1}([a(\sigma_{ij})]_s)] = \sum_{j=1}^{n} [a^{-1}([a(\sigma_{ij})]_s)]$$

Let $j = i$. By the condition 4^0 there exists $h \in H_s$ such that $[a^{-1}ha(e_i)]_r = [a^{-1}([a(e_i)]_s)]_r$ for every $r \neq i$. By Lemma 5.1 we have $[a^{-1}([a(e_i)]_s)]_r \leqslant \sigma_{ir}$ (for every r).

Let $j \neq i$. We take an arbitrary $x \leqslant e_j$ such that $H_{ij}(x) \cap F \neq \varnothing$. Let $b \in H_{ij}(x) \cap F$. Then by the Corollary 1(a) to Lemma 4.4 $b(e_i) + e_i = e_i + x$, whence

$$ab(e_i) + a(e_i) = a(e_i) + a(x) \qquad (\star)$$

We show that $[a^{-1}([ab(e_i)]_s)] + [a^{-1}([a(e_i)]_s)] \leqslant (\sigma_{i1}, \ldots, \sigma_{in})$ for every s.

We have already proved that $[a^{-1}([a(e_i)]_s)]_r \leqslant \sigma_{ir}$. Since $ab \in F$, we have $[(ab)^{-1}([ab(e_i)]_s)]_r = [b^{-1}a^{-1}([ab(e_i)]_s)]_r \leqslant \sigma_{ir}$. It is easy to verify that $[a^{-1}([ab(e_i)]_s)]_r \leqslant \sigma_{ir}$.

It follows from $(\star)$ that $[a(x)]_s \leqslant [ab(e_i)]_s + [a(e_i)]_s$, whence $[a^{-1}([a(x)]_s)] \leqslant (\sigma_{i1}, \ldots, \sigma_{in})$.

To complete the proof it remains to recall the definition of σ_{ij}.

Theorem 6.2. $\sum\limits_{j=1}^{n} \sigma_{ij} \in \overline{L_0(F)}$ *for every* i.

Proof. Let $a \in F$. We denote $u = \sum\limits_{j=1}^{n}\sum\limits_{s=1}^{n}[a(\sigma_{ij})]_s = \sum\limits_{s=1}^{n} u_s$, where $u_s = \sum\limits_{j=1}^{n}[a(\sigma_{ij})]_s$.

It is clear that $a(\sum\limits_{j=1}^{n}\sigma_{ij}) \leqslant u$. On the other hand, by Lemma 6.1 $a^{-1}(\sum\limits_{s=1}^{n} u_s) \leqslant \sum\limits_{j=1}^{n}\sigma_{ij}$, whence $a(\sum\limits_{j=1}^{n}\sigma_{ij}) = u \in \overline{L}_0$, hence $\sum\limits_{j=1}^{n}\sigma_{ij} \in \overline{L_0(F)}$.

Corollary 1. $K \subseteq \overline{L_0(F)}$.

Remark. The lattice K may not coincide with the lattice $\overline{L_0(F)}$: if $F = H$, then $K = L_0$, but $\overline{L_0(H)} = \overline{L}_0$.

Corollary 2. $G(\overline{L_0(F)}) \leqslant G(K)$.

§ 7. Net collections in $L_0^{'}$

Definition. We call by a net collection in $L_0^{'}$ a collection of elements $\tau = (\tau_{ij})$, $i, j = 1, \dots, n$, such that the following properties are fulfilled (for every i, j, k):

1) $\tau_{ij} \leqslant e_j$,
2) $\tau_{ii} = e_i$,
3) $\tau_{ij} \in L_0^{'}$,
4) for every $g \in G$ the following assertions are equivalent:
   (*i*) $[g(e_i)]_j \leqslant \tau_{ij}$
   (*ii*) $[g(\tau_{ki})]_j \leqslant \tau_{kj}$

(note that in 4) the only nontrivial implications are $(i) \Rightarrow (ii)$ for the distinct i, j, k: see the condition 6^0).

Lemma 7.1. *If* $\tau^\alpha = (\tau_{ij}^\alpha)$ *is a net collection in* $L_0^{'}$ *for every* $\alpha \in I$, *then* $\tau^{'} = (\tau_{ij}^{'})$, *where* $\tau_{ij}^{'} = \prod\limits_{\alpha \in I} \tau_{ij}^\alpha$, *is also a net collection in* $L_0^{'}$.

Proof. Follows from the definition of a net collection.

For every net collection $\tau = (\tau_{ij})$ in $L_0^{'}$ we denote by K_τ the sublattice of $L_0^{'}$, generated by zero and elements $\sum\limits_{j=1}^{n}\tau_{ij}$, $i = 1, \dots, n$. Note that the assertion 4*b*) from the definition of a net collection is equivalent to $g \in G(K_\tau)$.

Theorem 7.2. $G(K_\tau) = \langle H, H_{ij}(x) :\ x \leqslant \tau_{ij},\ i \neq j\rangle$.

Proof. We denote $V = \langle H, H_{ij}(x) :\ x \leqslant \tau_{ij},\ i \neq j\rangle$. By the definition of a net collection $G(K_\tau) \geqslant V$. Further, let $g \in G(K_\tau)$.

A. For every i $g(\sum\limits_{j=1}^{n}\tau_{ij})=\sum\limits_{j=1}^{n}\tau_{ij}$, then $[g(\sum\limits_{j=1}^{n}\tau_{1j})]_1=e_1$. By the condition 5^0 there exists $t_1\in G$ with the following properties: $[gt_1(e_1)]_1=e_1$; $t_1(e_k)=e_k$ for $k\neq 1$; $[t_1(e_1)]_k\leqslant\tau_{1k}$. We show that $t_1\in V$.

By Lemma 5.2 there exists $\bar t_2\in H_{12}([t_1(e_1)]_2)\leqslant V$ such that $[\bar t_2t_1(e_1)]_2=0$. Evidently $[\bar t_2t_1(e_1)]_k=[t_1(e_1)]_k$ for $k\neq 2$. Continuing, we will find $\bar t_k\in H_{1k}([t_1(e_1)]_k)\leqslant V$, $k=3,\dots,n$ such that $\bar t_n\dots\bar t_2t_1(e_1)=e_1$, whence $\bar t_n\dots\bar t_2t_1\in H$, hence $t_1\in V$.

B. Using the line of reasoning as in the part A, we find $g_1\in V$ such that $g_1gt_1(e_1)=e_1$. By induction we get $g\in V$.

Lemma 7.3. *Let i,j,k be pairwise distinct, and $g\in H_{ij}(x)$, $H_{ki}(y)\neq\varnothing$. Then $H_{kj}([g(y)]_j)\cap\langle H,H_{ij}(x),H_{ki}(y)\rangle\neq\varnothing$.*

Proof. Direct consequence of Lemmas 4.4, 5.2 and 5.3.

Theorem 7.4. *$\sigma=\sigma(F)$ is a net collection in $L_0^{'}$.*

Proof. By Lemma 5.5 $\sigma_{ij}\in L_0^{'}$. It remains to prove that it follows from $[g(e_i)]_j\leqslant\sigma_{ij}$ that $[g(\sigma_{ki})]_j\leqslant\sigma_{kj}$ (for the distinct i,j,k).

Let $y\leqslant\sigma_{ki}$ be such that $H_{ki}(y)\neq\varnothing$. By the condition 8^0 we can find $f\in H_{ij}([g(e_i)]_j)\subseteq F$ such that $[f(y)]_j=[g(y)]_j$. Then by Lemma 7.3 $[g(y)]_j\leqslant\sigma_{kj}$.

If $\sigma=\sigma(F)$, then we denote $K=K(F)=K_\sigma$.

Corollary 1. $G(K)=\langle H,H_{ij}(x)\cap F:\ x\leqslant e_j,\ i\neq j\rangle$.

Corollary 2. $G(K)\leqslant F$.

Corollary 3. $G(\overline{L_0(F)})\leqslant F$.

Corollary 4. *The groups $G(K)$ and F have the same transvections.*

Lemma 7.5. *The groups $G(\overline{L_0(F)})$ and F have the same transvections.*

Proof. Let $t\in H_{kl}(x)\cap F$, $\sum\limits_{i=1}^{n}x_i\in\overline{L_0(F)}$. Since $t\in F$, we have $t(\sum\limits_{i=1}^{n}x_i)=\sum\limits_{i=1}^{n}y_i\in\overline{L}_0$. It is easy to check that $x_i=y_i$ for $i\neq l$ and $x_l\leqslant y_l$. We obtain from the equality of dimensions that $\sum\limits_{i=1}^{n}x_i=\sum\limits_{i=1}^{n}y_i$.

Lemma 7.6. *If $\tau=(\tau_{ij})$ is a net collection in $L_0^{'}$, then $\tau=\sigma(G(K_\tau))$.*

Proof. If $t\in H_{ij}(x)\cap G(K_\tau)$, then $x\leqslant\tau_{ij}$. Thus $\sigma_{ij}\leqslant\tau_{ij}$, but the condition 7^0 implies that in fact we obtain the equality.

Thus, every net collection in $L_0^{'}$ consists of "the ideals of transvections".

§ 8. Proof of the main result

Lemma 8.1. $G(\overline{L_0(F)}) \trianglelefteq F$.

Proof. Let $g \in G(\overline{L_0(F)})$, $f \in F$, $l \in \overline{L_0(F)}$. We have $f^{-1}gf(l) = f^{-1}f(l) = l$ (since $f(l) \in \overline{L_0(F)}$), whence $f^{-1}gf \in G(\overline{L_0(F)})$.

Note that the groups $G(\overline{L_0(F)})$ and $G(K)$ may not coincide (for $m \neq 1$, see § 9). If $G(\overline{L_0(F)}) \geqslant H$, then by Lemma 7.5 $G(K) = G(\overline{L_0(F)})$, therefore the relation $G(K) \trianglelefteq F$ clearly holds true. But general case requires a special proof.

Theorem 8.2. $G(K) \trianglelefteq F$.

Proof. Let $f \in F$, $h \in H_t$. We put $g = f^{-1}hf$. Since by Lemma 5.1 $[g(e_i)]_j \leqslant \sigma_{ij}$ for every i, j, we obtain $g \in G(K)$ by Theorem 7.4. Due to Corollary 1 to Theorem 7.4, Lemma 5.3, and the equality $G(\overline{L}_0) = H_{12}(0)$, for the completion of the proof it remains to show that for every $f \in F$, $t \in H_{ij}(x) \cap F$, the automorphism $f^{-1}tf$ belongs to $G(K)$. It follows from Lemmas 7.5 and 8.1 that $t \in G(\overline{L_0(F)}) \trianglelefteq F$. Taking into account the inequality $G(\overline{L_0(F)}) \leqslant G(K)$ completes the proof.

Lemma 8.3. *If the lattice $\overline{L}_0(H)$ is finite, then the index of the subgroup $G(K)$ in the group F is finite.*

Proof. Since $f^{-1}hf \in G(K)$ for every $f \in F$ and $h \in H$, we have $hf(\sum_{j=1}^{n} \sigma_{ij}) = f(\sum_{j=1}^{n} \sigma_{ij})$. Hence $f(\sum_{j=1}^{n} \sigma_{ij}) \in \overline{L}_0(H)$, and subject to the settings of Lemma there is only a finite number of the possibilities for the values $f(\sum_{j=1}^{n} \sigma_{ij})$. The rest is clear.

Thus, Theorem 2.1 is proved completely.

Lemma 8.4. *Let M be a sublattice of $\overline{L}_0$. If $t \in H_{ij}(x) \cap \mathcal{N}_G G(M)$, then $t \in G(M)$.*

Proof. Let $x \in M$. Then $x = \sum_{k=1}^{n} x_k$, $x_k \leqslant e_k$. By the condition 4^0 there exists $h \in H_j \cap G(\overline{L}_0)$ such that $[t^{-1}ht(x_i)]_j = [t(x_i)]_j$. By the settings of Lemma $t^{-1}ht(x) = x$, whence $[t(x_i)]_j \leqslant x_j$. Hence $t(x) = x$.

§ 9. Application to linear groups

The description of subgroups in the general linear group $G = \mathrm{GL}(n, R)$ over a semilocal ring R, containing the group of diagonal matrices $D = \mathrm{D}(n, R)$, was obtained in [BV] in terms of nets over R.

This description consists in the following. Let R be a semilocal ring (that is, a ring, quotient of which modulo the Jacobson radical is Artinian), C its center (which is a commutative semilocal ring). Suppose that all fields of residues of C modulo its maximal ideals have at least seven elements. Then for every intermediate

subgroup F, $D \leqslant F \leqslant G$, there exists a unique D–net σ of order n over R such that $G(\sigma) \leqslant F \leqslant \mathcal{N}(\sigma)$, where $\mathcal{N}(\sigma)$ is the normalizer of $G(\sigma)$ in G.

We demonstrate how to deduce this description of the intermediate subgroups (frankly, in a slightly weaker form: instead of the semilocal rings we consider only the Artinian ones) from the results of §§ 4–8.

Let R be a right Artinian ring, all residue fields of center of which have at least seven elements, let $V = R^n$ be a free R-module of rank n, let $\bar{e}_1 = (1, 0, \dots, 0), \dots, \bar{e}_n = (0, \dots, 0, 1)$ be the canonical basis of V, $e_1 = \bar{e}_1 R, \dots, e_n = \bar{e}_n R$.

We denote by $L = L(V)$ the lattice of right submodules of the module V and by L_0 the sublattice of the lattice L generated by $e_1, \dots, e_n$. It is clear that L is a modular lattice of finite length, L_0 is a Boolean algebra.

Each element $g \in \mathrm{GL}(n, R)$ generates an automorphism of the lattice L. Namely, if ν is an element of L, then $g(\nu) = \{g(x) : \ x \in \nu\}$. Thus, one can assume that $G = \mathrm{GL}(n, R)$ and $H = G(L_0) = \mathrm{D}(n, R)$.

Using some elementary facts about the semilocal rings (see [Ba]; [Bo1]; [BV]), it is easy to verify that the conditions $1^0 - 10^0$ of Theorem 2.1 hold true. The assertion of the condition 11^0 is proved in [BV] (see the proof of Lemmas 3 and 4).

Thus, we can apply the results obtained in §§ 4–8.

Let F be an intermediate subgroup, $H \leqslant F \leqslant G$. Then $G(K) \trianglelefteq F$ for the lattice $K = K_\sigma$ (Theorem 8.2).

We put into correspondence to the lattice K the matrix of ideals in R $\sigma_K = (\sigma_{ji})$. Since $\sigma = (\sigma_{ij})$ is a net collection in $L_0^{'}$ (Theorem 7.4), we see that σ_K is a D–net of order n over R (see the definition of a net collection; note that the condition 3) from this definition implies that σ_{ij} is a two-sided ideal (see the papers cited above)).

Due to the condition 4) from the definition of a net collection the subgroup $G(K)$ coincides with the net subgroup $G(\sigma_K)$.

Thus we proved that for every subgroup F, $H \leqslant F \leqslant G$, there exists a D–net σ of order n over R such that

$$G(\sigma) \leqslant F \leqslant \mathcal{N}(\sigma), \tag{†}$$

where by $\mathcal{N}(\sigma)$ we denote the normalizer of $G(\sigma)$.

We show that a D–net σ which satisfies the condition (†) is uniquely determined.

Let σ_1 and σ_2 be two D–nets, which satisfy (†). Consider $K_1 = K(G(\sigma_1)), K_2 = K(G(\sigma_2))$. Since $G(K_1) = G(\sigma_1)$, $G(K_2) = G(\sigma_2)$, we have $G(K_1) \leqslant F \leqslant \mathcal{N}_G G(K_1), G(K_2) \leqslant F \leqslant \mathcal{N}_G G(K_2)$.

Each transvection containing in $G(K_1)$ belongs to $\mathcal{N}_G G(K_2)$ and, by Lemma 8.4, is contained in $G(K_2)$. The reverse statement is also true. Thus, $G(K_1)$ and $G(K_2)$ have the same transvections, whence $G(\sigma_1) = G(\sigma_2)$, therefore $\sigma_1 = \sigma_2$.

Thus we proved that for every subgroup F, $H \leqslant F \leqslant G$, there exists a unique D–net σ of order n over R such that $G(\sigma) \leqslant F \leqslant \mathcal{N}(\sigma)$, hence we obtained the required description of the intermediate subgroups.

If R is a left Artinian ring, then this description can also be obtained by passing to the opposite ring.

Remark 1. It follows from Lemma 8.3 that if the lattice $\overline{L}_0(H)$ is finite, then the index of $G(\sigma)$ in F is finite. The lattice $\overline{L}_0(H)$ consists in our case of the direct sums of two-sided ideals in R. So, if there is only a finite number of two-sided ideals in R (for example, if R is a semisimple Artinian ring), then $(F : G(\sigma)) < \infty$. (Indeed, a more powerful result is valid: see [BV]).

Remark 2. It was mentioned at the beginning of § 8 that the groups $G(\overline{L_0(F)})$ and $G(K)$ may not coincide. It is easy to construct examples of such phenomenon for the case of noncommutative R and $F = H$.

§ 10. Appendix

Fix $x \in \overline{L}_0 \cap L_0^{'}$ (write $x_k = [x]_k$) and a pair of indices i, j.

We say that $u \leqslant e_j$ satisfies the condition $(\triangle)$, if for every $f \in G$ it follows from the inequality $[f(e_i)]_j \leqslant u$ that $[f(x_i)]_j \leqslant x_j$.

Example. $u = 0$ satisfies the condition $(\triangle)$ for every x, i, j.

Lemma 10.1. *If u_1, u_2 satisfy the condition $(\triangle)$, then so does $u_1 + u_2$.*

Proof. By the condition 7^0 $u_k = \sum\limits_{l=1}^{s_k} y_{kl}$, where $H_{ij}(y_{kl}) \neq \varnothing,\ k = 1, 2$. By the condition 8^0 for every $f \in G$ there exists $g \in H_{ij}([f(e_i)]_j)$ such that $[g(x_i)]_j = [f(x_i)]_j$, and we can assume that $f \in H_{ij}(y)$ and $[f(e_i)]_j \leqslant u_1 + u_2$.

By the condition 10^0 $H_{ij}(y) \subseteq \langle H, H_{ij}(y_{kl}),\ k = 1, 2,\ l = 1, \ldots, s_k \rangle$.

Since $x_s \in L_0^{'}$, we see that $h(x_s) = x_s$ for every $h \in H$ and s.

If $t \in H_{ij}(y_{kl})$, then $[t(e_i)]_j = y_{kl} \leqslant u_k$, whence $t(x_i) \leqslant x_i + x_j$.

Corollary. *For every $x \in \overline{L}_0 \cap L_0^{'}$ and indices i, j there exists the maximal element $\tau_{ij} = \tau_{ij}(x) \leqslant e_j$, which satisfies the condition $(\triangle)$.*

Lemma 10.2. *$\tau = (\tau_{ij})$ is a net collection in $L_0^{'}$.*

Proof. We verify the conditions $1) - 4)$ from the definition of a net collection.

1) Clear.

2) Follows from the condition 6^0.

3) It is clear that for every $h \in H$ $h(\tau_{ij})$ satisfies the condition $(\triangle)$. Then by the definition of τ_{ij} we obtain $h(\tau_{ij}) \leqslant \tau_{ij}$.

4) By the condition 7^0 $\tau_{ki} = \sum\limits_{r=1}^{s} y_r$, where $H_{ki}(y_r) \neq \varnothing$. Let $g \in H_{ij}(z)$ and $f \in H_{kj}(y)$ be such that $[g(e_i)]_j \leqslant \tau_{ij}$ and $[f(e_k)]_j \leqslant \sum\limits_{r=1}^{s} [g(y_r)]_j$. We have to verify that $[f(x_k)]_j \leqslant x_j$.

By Lemma 7.3 for every r, $1 \leqslant r \leqslant s$, there exists $t_r \in H_{kj}([g(y_r)]_j) \cap \langle H, H_{ij}(z), H_{ki}(y_r) \rangle$.

Since $y_r \leqslant \tau_{ki}$ and $z \leqslant \tau_{ij}$, for every $t \in H_{ki}(y_r)$, $\bar{t} \in H_{ij}(z)$ we have $t(x_k + x_i + x_j) \leqslant x_k + x_i + x_j$ and $\bar{t}(x_k + x_i + x_j) \leqslant x_k + x_i + x_j$. Hence $[t_r(x_k)]_j \leqslant x_j$.

Further, by the condition 10^0 $H_{kj}(y) \subseteq \langle H, t_r,\ r = 1, \dots, s\rangle$, therefore $[f(x_k)]_j \leqslant x_j$. Lemma is proved.

Lemma 10.3. *If $g \in G$ and $[g(x_i)]_j \leqslant x_j$, then $[g(e_i)]_j \leqslant \tau_{ij}$.*

Proof. If $f \in G$ and $[f(e_i)]_j \leqslant [g(e_i)]_j$, then by the condition 6^0 $[f(x_i)]_j \leqslant [g(x_i)]_j \leqslant x_j$, therefore $[g(e_i)]_j$ satisfies the condition $(\triangle)$.

We introduce an equivalence relation on the set of sublattices of $L_0^{'}$, namely, we put $M_1 \sim M_2$, if $G(M_1) = G(M_2)$.

Theorem 10.4. *Provided that the conditions $1^0 - 11^0$ of Theorem 2.1 and the following condition*

12^0. $L_0^{'} \subseteq \overline{L}_0$

are fulfilled, for every subgroup F, $H \leqslant F \leqslant G$, there exists a unique class of the equivalent sublattices of $L_0^{'}$ such that $G(M) \trianglelefteq F$ for every M of this class.

Proof. Existence follows from Theorem 2.1.

Let M be a sublattice of $L_0^{'} \subseteq \overline{L}_0$. By Lemma 10.2 $\tau(x) = (\tau_{ij}(x))$ is a net collection in $L_0^{'}$ for every $x \in M$. We put $\tau_{ij}^{'} = \prod\limits_{x \in M} \tau_{ij}(x)$. By Lemma 7.1 $\tau^{'} = (\tau_{ij}^{'})$ is also a net collection in $L_0^{'}$. By Theorem 7.2 $G(K_{\tau'})$ is generated by H and its transvections.

We show that $G(K_{\tau'}) = G(M)$. Indeed, let $g \in G(K_{\tau'})$. Then $[g(e_i)]_j \leqslant \tau_{ij}^{'}$ for every i, j, therefore $[g(e_i)]_j \leqslant \tau_{ij}(x)$ for every $x \in M$ and hence $g \in G(M)$.

If $g \in G(M)$, then $[g(x_i)]_j \leqslant x_j$ for every $x \in M$, and by Lemma 10.3 $[g(e_i)]_j \leqslant \tau_{ij}(x)$, whence $[g(e_i)]_j \leqslant \tau_{ij}^{'}$, therefore $g \in G(K_{\tau'})$.

The uniqueness follows now from Lemma 8.4.

It follows from Theorem 10.4 that all closed subgroups in $\mathfrak{N}$ (see § 1) are of the form $G(K_\tau)$, where τ belongs to the set of net collections in $L_0^{'}$.

Further, the sublattices $L_0^{'}(\tau) = L_0^{'}(G(K_\tau)) = \{x \in L_0^{'} :\ g(x) = x$ for every $g \in G$ such that $[g(e_i)]_j \leqslant \tau_{ij}$ for every $i, j\}$ exhaust all closed sublattices in $\mathfrak{M}$.

Thus, the Galois correspondence introduced in § 1 is a bijection between the set of subgroups of the form $G(K_\tau)$ and the set of sublattices of the form $L_0^{'}(\tau)$ (both sets are in one-to-one correspondence with the set of net collections in $L_0^{'}$, see Lemma 7.6).

We consider now the case $m = 1$ in more detail. Note that the condition 12^0 of Theorem 10.4 is automatically fulfilled in the settings of Theorem 3.1 (since L_0 is a Boolean algebra).

It was proved in [S] that all sublattices of $L_0^{'} = L_0$, containing 0 and 1, are closed, then each class of the equivalent sublattices of L_0, containing 0 and 1, consists of one element, therefore we obtain uniqueness stated in Theorem 3.1. Note also that it follows from $G(\overline{L_0(F)}) = G(K)$ (see the beginning of § 8) that $K = \overline{L_0(F)}$.

Let now L, L_0 and G be as in § 9. It is easy to prove that in that case the condition 12^0 of Theorem 10.4 is fulfilled. Hence we obtain

Theorem 10.5. *For every intermediate subgroup F, $\mathrm{D}(n,R) \leqslant F \leqslant \mathrm{GL}(n,R)$, there exists a unique class of the equivalent sublattices of $L_0^{'}$ such that $G(M) \trianglelefteq F$ for every M of this class.*

It is clear that there is a one-to-one correspondence between net collections in $L_0^{'}$ and nets of order n over R.

Further, closed subgroups in $\mathfrak{N}$ are exhausted by the net subgroups, and closed sublattices in $\mathfrak{M}$ by the sublattices $L_0^{'}(\sigma) = \{x \in L_0^{'} : \sigma_{ij}x_j \subseteq x_i\}$, σ is a net.

It is easy to construct examples showing that a lattice M such that $G(M) \trianglelefteq F$ is not uniquely defined. It is clear that every class of the equivalent sublattices of $L_0^{'}$ contains the maximal element, which is a closed sublattice, and the "canonical sublattice", generated by zero and sums of "the ideals of transvections". Usually the "canonical" one is rather far from its closure, but it is possible to construct examples showing that this sublattice is not the minimal element of the class.

Final remarks

1^o. We are sure that Theorem 2.1 can be generalized to a class of complete modular lattices (not necessary of finite length), and the description of the intermediate subgroups of the general linear group over a semilocal ring containing the group of diagonal matrices can be obtained as in § 9.

2^o. There is a strong analogy between our results and results of N.A.Vavilov [V5] on the geometry of tori, which is not completely understood at this moment.

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